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RECENT RESEARCH ON THE PROBLEM OF BASE PRESSURE

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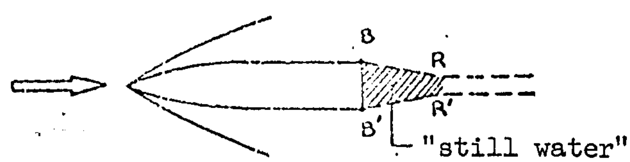
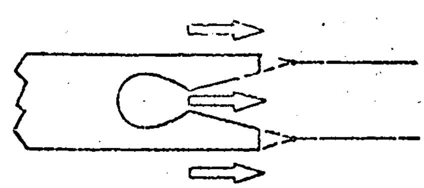
Introduction

Consider an artillery projectile in the shape of an ogive and cylindrical body terminating abruptly in a flat base at BB'. The supersonic flow around this projectile breaks away at BB' and merges downstream at RR', thus containing in the empty zone a "still water", in which the pressure is almost uniform and is lower than the ambient pressure. This results in a drag which sometimes represents from 30 to 40 percent of the total projectile drag. /2



The same problem exists at the exit of an aircraft or rocket reactor, whenever the exit area of the propulsive nozzle is less than the total area of the base surface. Again in this case the internal and external streams break away at the level of the base section and merge further downstream, thereby creating between the two streams a zone labelled "still water". The pressure in this zone is part of the propulsive force which becomes greater as the base area upon which it applies becomes greater.



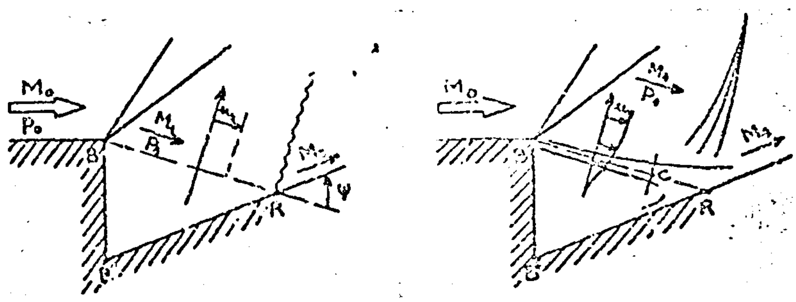
A few years ago the solution to these problems of base drag was treated in a purely empirical way, but recent research, which was supported in part by the O.N.E.R.A., has given us a clear and efficient method of treating these problems (Refs. 1, 2, 3). The object of the present paper is to discuss a few basic problems connected with this theory.

1. Principle of the Method /3

First let us take a very simple mathematical case, that of a two-dimensional uniform supersonic flow leaving the edge BB' at B. It undergoes a Meyer expansion, characterized by a deflection ψ_1 and by a Mach number M_1 whose values are related to Mach M_0 and to the ratio of the pressure upstream p_0 to the pressure p_1 existing at the still water zone.

First assume a perfect flow and that the pressure at the still water p_1 is given, with $p_1 < p_0$ for example. The flow (M_1) remains uniform until an impact at R with the downstream wall. At R we have a shock whose intensity is related to Mach M_1 and to the deflection ψ undergone by the stream at this place.

Now observe that in the real flow conditions, i.e., for a viscous fluid, the discontinuity in velocity $c - u_1$ between the two parts separated by the jet line BR cannot remain the same, and changes progressively into a continuous variation of the velocity, with the velocity almost zero in the still water and equal to u_1 outside it. The flow region which is affected by this velocity change is labelled the "mixture layer" and the corresponding phenomenon or effect is called the "mixture phenomenon". This effect can be considered isobaric, provided the departure height BB' is sufficiently great as compared to the thickness of the mixture layer.



A. perfect fluid B. real fluid

Now examine the behavior of the fluid particles which traverse the mixture zone and reach the merger zone. Since maintenance of the steady-state regime demands a conservation of the fluid mass contained in the still water, the streamline (j) emanating exactly from B must reach R. Any streamline situated above (j) must extend downstream from R, and any streamline situated below (j) must return to the still water. The back-flow which is immediately established upstream from R means that there exists an opposing pressure gradient; in other words, in the flow of the nonviscous fluid there exists a continuous compressional wave, as /4 indicated in Figure B above. This wave replaces the shock wave of Figure A relative to the perfect fluid.

Since R is a point of rest on (j), the available momentum on (j) at point C, immediately ahead of the recompression zone, must be of such value as determined by the condition that it must go to zero at R due to the opposing pressure gradient and to the viscous forces met from C to R.

If this condition is satisfied, it is evident that the fluid particles moving below (j) are animated with a lower velocity, and make a U-turn before reaching R; whereas the particles moving above (j) are faster and will be able to overcome the braking action from C to R and escape downstream.

The two basic problems which stem from this analysis are therefore the following:

1) To determine the flow conditions in the mixture zone. We shall characterize this flow at any point by the variables

$$\varphi = \frac{u}{u_1} \qquad\qquad \theta = \frac{\rho}{\rho_1}$$

The corresponding problem is called the "isobaric mixture problem".

2) Knowing the profiles $\varphi(y)$ and $\theta(y)$ in the mixture layer, near C, where the merger effect starts to take place, to determine which streamline (1) will reach rest point R, with an angle ψ of deflection of the flow (M_1). This problem will be called the "merger problem". We shall not discuss it here. It will suffice to assume, in order to understand the rest of this discussion, that angle ψ is a known function of Mach M_1 and of the variables φ_1 and θ_1 which characterize the state of the flow immediately upstream from the recompression zone, on the boundary streamline (1).

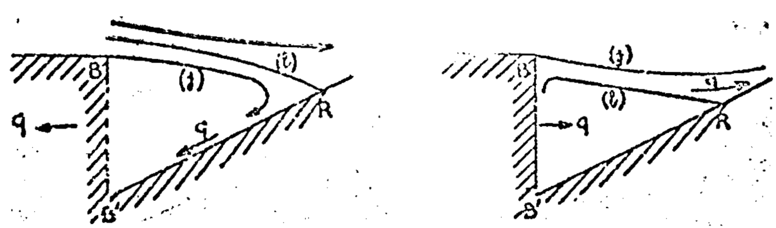
We shall take this function as /5

$$\psi = \psi(\varphi_1, \theta_1, M_1) \tag{1}$$

and term it the angular criterion for merger. Since φ_1 and θ_1 are given by the theory of the mixture as functions of a reduced ordinate η_1, it is clear that ψ is an increasing function of η_1. Indeed, the farther boundary line (l) lies from the still-water zone, the greater the momentum, and consequently, the greater the recompression it can overcome. This recompression is itself determined in the nonviscous flow to be an increasing function of defection angle ψ, for a given Mach number M_1.

Suppose for example that (M_1) and ψ are given. Equation (1) yields the value of η_1, i.e., the boundary flow line (l).

- If (l) and (j) coincide, the conservation condition is met and M_1 and ψ are compatible.
- If (l) is above (j), a certain quantity of fluid $q = \rho_1 u_1 \int_j^l \theta\varphi dy > 0$ originating from the outer flow will enter the still water for every unit of span of the device. Then ψ and M_1 will, in the steady-state regime, be compatible only if an equivalent suction of q in the still water is possible.
- If (l) is below (j), quantity q changes sign. This quantity is constantly taken from the still water and ejected downstream of R. Also, ψ and M_1 will be compatible only if an injection of q into the still water is possible.



If this injection or this suction into or from the still water are possible without modifying the behavior of the stream in the mixture layer (in other words, without perturbing functions φ and θ, which represent the mixture layer), then there exists a one-to-one correspondence between q and (φ_1, θ_1) so that Equation (1) can be written in the form /6

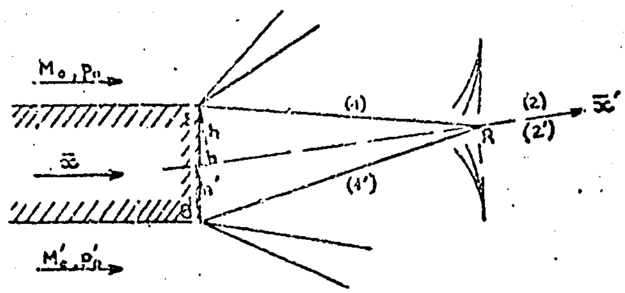
$$\psi = \psi\ (M_1, C_Q) \qquad (1')$$

where $C_q = \dfrac{q}{\rho_1 u_1 h}$ (h being the reference length, for example BB').

C_q is $>$ 0 or $<$ 0 depending on the case.

Before examining in detail some of the problems which arise from this analysis, let us assume them to be solved and rapidly examine the method to follow in the discussion of the problem of the base pressure.

Consider for example the case of two uniform two-dimensional streams (M_0) and (M'_0) coming from either side of a base BB' and meeting.

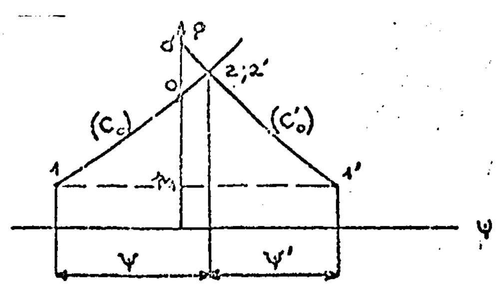


Take a base pressure p_1. The geometry of figure BB'R is immediately determined by the Meyer expansions at B and B'.

$$\widehat{BR_1 x} = f\left(M_0, \frac{p_1}{p_0}\right) \longrightarrow M_1$$

$$\widehat{B'R_1 \vec{x}} = -f\left(M'_0, \frac{p_1}{p'_0}\right) \longrightarrow M'_1$$

The final direction $\overleftrightarrow{Rx}'$, common to both flows (2) and (2') after merger, results from the two isentropic compressions (1) → (2) and (1') → (2'), with $p_2 = p_{2'}$. The standard solution to these problems is shown schematically by the pressure deflection diagram below: /7



(C_0) and (C'_0) are curves for the Prandtl-Meyer expansion or isentropic recompression of the upstream flows (C) and (C').

(Note that the upstream state (2) (2') does not depend here on the assumption made for p_1).

The diagram associates to each value of p_1 two deflections at R

$$\psi = \widehat{BR, Rx'} \qquad \psi' = \widehat{B'R, Rx}$$

Using the merger criterion for each of the two flows (1) and (1'), we have

$$\psi = \psi\ (M_1,\ C_q)$$

$$\psi' = \psi\ (M'_1,\ C'_q)$$

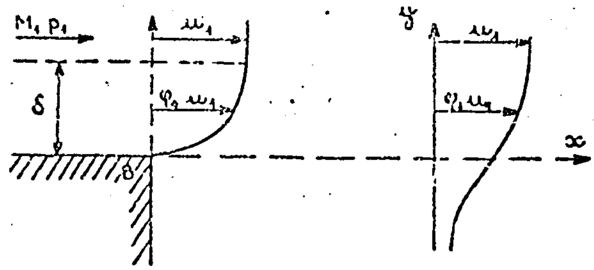
From this are deduced C_q and C'_q, or the quantities $q = \rho_1\ u_1\ h\ C_q$ and $q' = \rho'_1\ u'_1\ h'\ C'_q$ that the two flows (1) and (1') pour into the still water, respectively. The value chosen for p_1 will be suitable if it leads to $q + q' = 0$; in other words, if one of the flows pours into the still water a quantity equal to the quantity which the other removes. The solution is therefore obtained by trial and error with p_1.

Having reviewed the general method, we will now examine some of the fundamental problems to be solved, and we shall restrict our attention to the case of a turbulent mixture, since this is a very important practical case. /8

2. Problem of the Isobaric Turbulent Mixture

As we have seen, this problem is to describe a layer of mixture flowing $(M_1,\ p_1)$ out of B and into an infinite atmosphere, with a pressure p_1.

If the initial stream is perfectly uniform, the problem becomes the standard one (Gortler, Ref. 4). But if there exists an initial boundary layer upstream from B, the problem becomes difficult. An approximate solution to it has been given by H. Korst (Ref. 5).



The principle involved in this solution is to simplify the equation for the momenta belonging to the viscous layer so as to bring it to the form of the heat equation. In this way, we can obtain for the distribution of velocities a solution which satisfies exactly the boundary conditions, and also an approximate solution to the momentum equation. The temperature distribution is immediately deduced from the velocity distribution by making an assumption on the turbulent Prandtl number. These two distributions will finally be suitably shifted parallel to the y axis in order to satisfy the momentum theorem in a general way at each abscissa.

The Relation Between the Temperature Profile and the Velocity Profile /9

We shall first recall a standard result of the theory of the boundary layer. The temperature profile is deduced from the velocity profile by making certain assumptions. For this we write the equations for the momenta and the energy of the isobaric mixture layer. The first equation is written

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\varepsilon_t \frac{\partial u}{\partial y}\right) \qquad (1)$$

In this expression, ε_t is a turbulent transport coefficient analogous to the viscosity coefficient of the laminar regime. We shall not determine its form for the moment.

As a second unknown function, we take the total enthalpy H_i

$$H_i = H + \frac{u^2}{2} \qquad (v \ll u)$$

and we write the energy equation:

$$\rho u \frac{\partial H_i}{\partial x} + \rho v \frac{\partial H_i}{\partial y} = \frac{\partial}{\partial y}\left(\varepsilon_t u \frac{\partial u}{\partial y} - q\right)$$

In this expression q represents the transverse heat flux which by analogy with the equivalent expression of the laminar regime, is written for a thermally perfect gas:

$$q = -\frac{\lambda_t}{C_p}\frac{\partial H}{\partial y} \qquad (dH = C_p\, dT)$$

where λ_t is a turbulent coefficient of conductivity. We introduce, just as in the laminar case, a turbulent Prandtl number defined by

$$P_{r_t} = \frac{\mu_t C_p}{\lambda_t}$$

and we assume this coefficient (number) to be equal to one. This assumption is a priori an arbitrary one which must be tested by its own consequences.

The energy equation then becomes: /10

$$\rho u \frac{\partial H_i}{\partial x} + \rho v \frac{\partial H_i}{\partial y} = \frac{\partial}{\partial y}\left(\varepsilon_t \frac{\partial H_i}{\partial y}\right) \qquad (2)$$

In this form we can recognize from the analogies of (1) and (2) that to every solution u (x, y) of (1) there corresponds a solution of (2) in the form

$$H_i\ (x,\ y) = \alpha\ u + \beta$$

where α and β are constants to be determined from the boundary conditions.

In particular, if H_m is the enthalpy of rest in the still water and if H_{i_1} is the enthalpy of rest in the outer stream, we have:

$$H_i(x, y) = H_m + (H_{i_1} - H_m)\,\varphi \qquad (3)$$

since $\varphi = 0$ in the still water,
$\varphi = 1$ outside.

Restricting ourselves to the case of perfect gases ($H = C_p\ T$), we can write, neglecting $v^2 \ll u^2$,

$$H_i = H + \frac{u^2}{2} = C_p\ T + \frac{u^2}{2}$$

from which the law of temperature distribution is:

$$\frac{1}{\theta} = \frac{T}{T_1} = \frac{T_m}{T_1} + \left(\frac{T_{i_1}}{T_1} - \frac{T_m}{T_1}\right)\varphi - \frac{\gamma-1}{2} M_1^2 \varphi^2 \qquad (4)$$

A very simple special case occurs when $T_m = T_{i_1}$. Then from (3), H_i and T_i are constant in the mixture layer and (4) becomes:

$$\frac{1}{\theta} = \frac{T_{i_1}}{T_1} - \frac{\gamma-1}{2} M_1^2 \varphi^2 \qquad (4')$$

Therefore, as soon as the function $\varphi(\eta)$ is found, (4) and (4') will give the corresponding distribution $\theta(\eta)$. That function will generally depend on the Mach number M_1 and the ratio $\frac{T_m}{T_i}$.

Velocity Profile /11

The momentum equation for the isobaric mixture layer is written:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial \tau}{\partial y}$$

where τ is the turbulent friction.

It is easy to determine that the second term of the left-hand side is negligible with respect to the first one. The equation is simplified by writing, analogously with the laminar case,

$$\frac{\partial \tau}{\partial y} = \rho e \cdot \frac{\partial^2 u}{\partial y^2}$$

where e is a function of x characterizing the strength of the turbulent mixture at that abscissa. In the momentum equation thus simplified, i.e.,

$$u \frac{\partial u}{\partial x} = e \frac{\partial^2 u}{\partial y^2}$$

we introduce one final simplification by replacing u in the left-hand side with an average value over the mixture zone, for example $\frac{u_1}{2}$ (where u_1 is the outside velocity).

The equation finally becomes:

$$\frac{\partial u}{\partial x} = \frac{2c(x)}{u_1} \frac{\partial^2 u}{\partial y^2}$$

This equation can be put into the form of the heat equation by the following change of variables

$$\xi = \int_0^x \frac{2c(x)}{u_1} dx \qquad \varphi = \frac{u}{u_1}$$

It becomes:

$$\boxed{\frac{\partial \varphi}{\partial \xi} = \frac{\partial^2 \varphi}{\partial y^2}} \tag{5}$$

where the integral is a well-known elementary one, giving:

$$\varphi^*(y,\xi) = \frac{1}{2\sqrt{\pi\xi}} e^{-y^2/4\xi}$$

We must find a solution to (5) capable of satisfying the following conditions:

a) At $x = 0$ ($\xi = 0$) the velocity profile is generally, because of the presence of the boundary layer, of the form: /12

$$\xi = 0 \begin{cases} y > \delta & \varphi = 1 \\ \delta > y > 0 & \varphi = \varphi_0\left(\frac{y}{\delta}\right) \\ y < 0 & \varphi = 0 \end{cases}$$

b) At any positive abscissa, the velocity for $y = +\infty$ is $u = u_1$, whereas for $y = -\infty$ it is $u = 0$, from which:

$$\begin{array}{lll} \xi > 0 & y = +\infty & \varphi = 1 \\ \xi > 0 & y = -\infty & \varphi = 0 \end{array}$$

A solution which satisfies these conditions is obtained by superpositions of the elementary solution of the form:

$$\varphi(y,\xi) = \frac{1}{\sqrt{\pi}} \int_0^\infty \varphi_0\left(\frac{\alpha}{\delta}\right) e^{-\left(\frac{\alpha - y}{4\xi}\right)^2} \frac{d\alpha}{2\sqrt{\xi}} \tag{6}$$

It can be verified that this solution satisfies the boundary conditions a) and b) above.

Conditions a)

Take:

$$\frac{\alpha - y}{2\sqrt{\xi}} = \beta$$

from which:

$$\varphi = \frac{1}{\sqrt{\pi}} \int_{-y/2\sqrt{\xi}}^{\infty} \varphi_0 \left(\frac{y + 2\beta\sqrt{\xi}}{\delta}\right) e^{-\beta^2} d\beta$$

When ξ tends to zero the lower boundary of the integral tends to $-\infty$, as as $\xi^{-1/2}$. Separate the integral into three parts

$$\varphi = \frac{1}{\sqrt{\pi}} \int_{-Y}^{+Y} \varphi_0 \left(\frac{y + 2\beta\sqrt{\xi}}{\delta}\right) e^{-\beta^2} d\beta + \frac{1}{\sqrt{\pi}} \int_{-\frac{y}{2\sqrt{\xi}}}^{-Y} \varphi_0 \left(\frac{y + 2\beta\sqrt{\xi}}{\delta}\right) e^{-\beta^2} d\beta + \frac{1}{\sqrt{\pi}} \int_{Y}^{+\infty} \varphi_0 \, e^{-\beta^2} d\beta$$

with, for example, $Y = \frac{y}{\varepsilon} \xi^{-1/4}$.

Since φ_0 is bounded ($\varphi_0 \leqslant 1$), the last two integrals obviously tend to zero, since their upper and lower boundaries increase infinitely in absolute value and since

$$\int_{-\infty}^{+\infty} e^{-\beta^2} d\beta$$

is a convergent integral.

In respect to the first integral, note that for /13

$$-\frac{y}{2} \xi^{-1/4} < \beta < +\frac{y}{2} \xi^{-1/4}$$

the value of $2\beta\sqrt{\xi}$ tends to zero with ξ, so that in this region we can write $\varphi_0 \left(\frac{y + 2\beta\sqrt{\xi}}{\delta}\right) = \varphi_0 \left(\frac{y}{\delta}\right)$ at the boundary $\xi = 0$.

We have, finally, at the boundary:

$$\varphi = \frac{\varphi_0\left(\frac{y}{\delta}\right)}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\beta^2}\, d\beta = \varphi_0\left(\frac{y}{\delta}\right)$$

Conditions b)

Taking any abscissa $\xi > 0$, let us verify whether, when y tends to $+\infty$, φ tends to 1, and when y tends to $-\infty$, whether φ tends to zero.

Take:

$$\frac{y}{2\sqrt{\xi}} = \eta \qquad \frac{\delta}{2\sqrt{\xi}} = \eta_0 \qquad \frac{\alpha}{2\sqrt{\xi}} = \lambda \tag{7}$$

Then the proposed solution is written:

$$\varphi(\eta) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \varphi_0\left(\frac{\lambda}{\eta_0}\right) e^{-(\lambda-\eta)^2}\, d\lambda$$

Since $\varphi_0 = 1$ for $\lambda > \eta_0$ (in other words, $y > \delta$), we will be able to separate $\varphi(\eta)$ in two parts:

$$\varphi(\eta) = \frac{1}{\sqrt{\pi}} \int_0^{\eta_0} \varphi_0\left(\frac{\lambda}{\eta_0}\right) e^{-(\lambda-\eta)^2}\, d\lambda + \frac{1}{\sqrt{\pi}} \int_{\eta_0}^{\infty} e^{-(\lambda-\eta)^2}\, d\lambda$$

This latter integral can be written, taking

$$\lambda - \eta = \beta$$

$$\frac{1}{\sqrt{\pi}} \int_{\eta_0}^{\infty} e^{-(\lambda-\eta)^2}\, d\lambda = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-\beta^2}\, d\beta - \frac{1}{\sqrt{\pi}} \int_0^{\eta_0-\eta} e^{-\beta^2}\, d\beta$$

The first of these integrals is equal to $\frac{1}{2}$ and the second one to /14

$$\frac{1}{2}\, \mathrm{erf}\,(\eta-\eta_0) = \frac{1}{\sqrt{\pi}} \int_0^{\eta-\eta_0} e^{-\beta^2}\, d\beta$$

from which, finally:

$$\boxed{\varphi = \frac{1}{2}\left[1+\mathrm{erf}(\eta-\eta_0)\right] + \frac{1}{\sqrt{\pi}}\int_0^{\eta_0} \varphi_0\left(\frac{\lambda}{\eta_0}\right) e^{-(\lambda-\eta)^2}\, d\lambda} \tag{8}$$

In this form we can immediately recognize with η_0 chosen that when η increases infinitely in absolute value, the first term tends to unity. The second term tends to zero, since for $\lambda < \eta_0$ this term is smaller than

$$\frac{e^{-(\eta-\eta_0)^2}}{\sqrt{\pi}}\int_0^{\eta_0} \varphi_0\, d\lambda$$

and this expression has a bounded integral and an exponential which tends to zero. When η tends to $-\infty$, the expression erf $(\eta - \eta_0)$ tends to -1.

The second term always tends to zero; therefore φ tends to zero.

Relations Between ξ and x

From experience with incompressible jets we are led to take for e(x) the expression

$$\frac{e(x)}{u_1} = \frac{x}{4\sigma^2}$$

so that, from (2)

$$\xi = \frac{x^2}{4\sigma^2}$$

from which

$$2\sqrt{\xi} = \frac{x}{\sigma}$$

The reduced coordinates η and η_0 introduced in (7) are written

$$\eta = \sigma\frac{y}{x} \qquad \eta_0 = \sigma\frac{\delta}{x} \tag{7'}$$

and solution (6) becomes:

$$\varphi(\eta) = \frac{1}{2}\left[1+\mathrm{erf}(\eta-\eta_0)\right] + \frac{1}{\sqrt{\pi}}\int_0^{\eta_0} \varphi_0\left(\frac{\lambda}{\eta_0}\right) e^{-(\lambda-\eta)^2}\, d\lambda \tag{8'}$$

The Influence of the Initial Boundary Layer on the Velocity Profile /15

In the absence of the initial boundary limit we have $\eta_0 = 0$, and (8') therefore takes the very simple form:

$$\overline{\varphi} = \frac{1}{2}(1 + \operatorname{erf} \eta). \qquad (9)$$

This form has been given by Gortler as a first order approximation to the unsimplified equations for the turbulent mixture.

By comparing (8') and (9) we immediately see that the velocity profile will be the less distorted by an initial boundary layer the smaller η_0 is, or in other words, the smaller $\frac{\delta}{x}$. This is what happens asymptotically when x increases to infinity.

In practice it suffices that $\frac{\delta}{x}$ be of the order of 1/10 in order for expression (9) to correctly represent the shape of the velocity profile.

3. Search for η_1 in the General Case

Consider the problem of an isobaric mixture characterized at the origin B by a uniform stream (p_1, M_1), extending in the neighborhood of the wall into a boundary layer having a profile $\frac{u}{u_1} = \varphi_0\left(\frac{y}{\delta}\right)$. Assume in addition that into the still water a quantity of mass q and momentum i (per unit of span) is injected.

Let us try to determine the boundary streamline (1) which, when merged, will satisfy the equilibrium condition of the still water.

As stated previously, we assume the problem of the mixture solved. We know in this way the shape of the velocity and temperature profiles:

$$\frac{u}{u_1} = \varphi(\eta) \qquad \frac{T_1}{T} = \frac{\rho}{\rho_1} = \theta(\eta)$$

with $\eta = \frac{\sigma y}{x}$ + constant.

The constant which enters in the expression for η will be so determined that the momentum condition be generally satisfied between the abscissa x = 0 and any other abscissa x.

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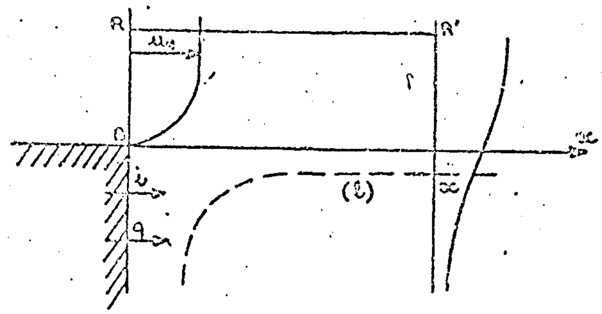


Let RR' be a streamline sufficiently far from Bx so that it lies outside of the viscous zone. RR' is parallel to Bx. Let us write the momentum theorem between x = 0 and x with $y \leqslant y_R$.

At the origin the input momentum is, per unit of span:

$$I_{(0)} = i + \int_0^{y_R} \rho u^2 \, dy$$

or, after dividing by $\rho_1 u_1^2$

$$\frac{I(0)}{\rho_1 u_1^2} = \frac{i}{\rho_1 u_1^2} + \int_0^{y_R} \frac{\rho u}{\rho_1 u_1^2} \, dy$$

At abscissa x the exit momentum is

$$I(x) = \int_{-\infty}^{y_{R'}} \rho u^2 \, dy$$

or, using the dimensionless variables:

$$\frac{I(x)}{\rho_1 u_1^2} = \frac{x}{\sigma} \int_{-\infty}^{\eta_{R'}} \theta \varphi^2 \, d\eta$$

Since this is an isobaric process we must have:

$$I_{(0)} = I(x)$$

from which the condition

$$\frac{\sigma}{x} \frac{i}{\rho_1 u_1^2} + \frac{\sigma}{x} \int_0^{y_R} \frac{\rho u^2}{\rho_1 u_1^2} \, dy = \int_{-\infty}^{\eta_{R'}} \theta \varphi^2 \, d\eta \qquad (1)$$

Since the left-hand side is assumed known, and since $\theta(\eta)$ and $\varphi(\eta)$ are determined, this relation defines the quantity η_A, corresponding to streamline (RR'). In other words, it defines the y shift of the velocity and temperature profiles, such that conservation of momentum be satisfied at abscissa x. /17

Now let us try to define boundary line (1). The necessary and sufficient condition for the fluid mass which is contained in the still water to be conserved is that between (1) and (RR') and at every abscissa we find a quantity equal to the quantity at location x = 0, i.e.,

$$q + \int_0^{y_R} \rho u \, dy = \int_{y_1(x)}^{y_{R'}} \rho u \, dy$$

or, using dimensionless variables

$$\frac{\sigma}{x} \frac{q}{\rho_1 u_1} + \frac{\sigma}{x} \int_0^{y_R} \frac{\rho u}{\rho_1 u_1} \, dy = \int_{\eta_1}^{\eta_{R'}} \theta \varphi \, d\eta \tag{2}$$

Taking (2) - (1) we have

$$\frac{\sigma}{x}\left(\frac{q}{\rho_1 u_1} - \frac{i}{\rho_1 u_1^2}\right) + \frac{\sigma}{x} \int_0^{y_R} \frac{\rho u}{\rho_1 u_1}\left(1 - \frac{u}{u_1}\right) dy = \int_{-\infty}^{\eta_{R'}} \theta \varphi (1-\varphi) \, d\eta - \int_{-\infty}^{\eta_1} \theta \varphi \, d\eta$$

Note that the two R-dependent integrals converge. We can therefore go to the limit $y_R \to \infty$. Under these conditions

$$\int_0^{\infty} \frac{\rho u}{\rho_1 u_1}\left(1 - \frac{u}{u_1}\right) dy = \delta^{**}$$

which is the momentum thickness for the initial boundary layer. The result sought is therefore written:

$$\boxed{\int_{-\infty}^{\eta_1} \theta \varphi \, d\eta = \frac{\sigma}{x}\left(\frac{i}{\rho_1 u_1^2} - \frac{q}{\rho_1 u_1} - \delta^{**}\right) + \int_{-\infty}^{+\infty} \theta \varphi (1-\varphi) \, d\eta} \tag{3}$$

In this expression, the right-hand side is known; the only unknown is therefore η_1 which, in this way, is determined, as are the results $\varphi(\eta_1)$ and $\theta(\eta_1)$, and consequently the merger criterion for the line (1).

Now let us consider the ideal case of a flow without initial boundary layer ($\delta^{**} = 0$) with no suction or ejection: $i = q = 0$, and with a uniform distribution of the total enthalpy ($T_m = T_{i_1}$). We term this case the reference case. We have then: /18

$$\varphi = \bar{\varphi}(\eta) = \frac{1}{2}(1 + \operatorname{erf} \eta)$$

$$\theta = \bar{\theta}(\eta) = \frac{1}{\dfrac{T_{i_1}}{T_1} - \dfrac{U^2}{\dfrac{2 c_p T_1}{M_1^2}}} = f(\eta, M_1)$$

Expression (3) shows then that $\eta_j = \eta_1$ and depends only on M_1. The same holds for the merger criterion in the reference case.

In the general case, we can assume that the injection does not greatly modify the mixture functions $\varphi(\eta)$ and $\theta(\eta)$, provided, however, that i and q have a fair value. We have on the other hand seen that if $\frac{x}{\delta_1}$ is fairly large, functions θ and φ depend very little on the presence of the initial boundary layer.

With these reservations in mind, we can consider that the integral of the right-hand side is a function of only M_1 and T_m/T_{i_1}. We can state that the following results:

a) With everything else being equal, the effect of an initial boundary layer having a momentum thickness δ^{**} on the merger criterion is the same as the effect of the injection into the still water at very low velocity of a quantity $\rho_1 \ u_1 \delta_1$.

b) The effect of an injection of unit mass q into the still water decreases η_1; in other words, it decreases the angle of merger (since ψ is an increasing function of η_1).

c) The effect of the injection of a momentum into the still water increases η_1, in other words it increases the angle of merger.

Since a standard injection entails practically a simultaneous injection of momentum:

$$q = \rho_j \cdot V_j \qquad f = \rho_j \cdot V_j^2$$

We see that since V_j increases from zero, the effect of an injection per unit mass will first be overwhelming, and ψ will decrease; then ψ will go through a minimum and increase.

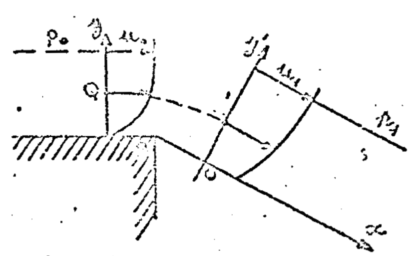
d) The merger criterion, such as the one described in Section 1, can be considered general, with the reservations indicated above, but we must take: /19

$$C_q = \frac{q}{\rho_1 u_1 h} + \frac{\delta^{**}}{h} - \frac{\dot{m}}{\rho_1 u_1 h}$$

4. Behavior of the Boundary Layer During the Expansion at B

The preceding calculations have shown the influence of the initial boundary layer upon the merger criterion. The question now is to determine in every special case the state of this boundary layer at the origin of the isobaric mixture, i.e., after the expansion at B.

More precisely, given a boundary layer with a velocity profile $\varphi = \varphi_0\left(\frac{y}{\delta}\right)$ before the expansion, the question is to calculate the profile $\varphi_1\left(\frac{y'}{\delta'}\right)$ of the boundary layer after the expansion.



In order to perform these calculations, with the details mentioned in Ref. 2, we shall make the following assumptions:

A) At a very small distance from B and downstream - of the order of δ - we can find a cross section Oy' where all the velocities have taken a direction parallel to Ox, and where the pressure has become uniform and equal to p_1.

B) If Q and Q' are two points situated on the same streamline before and after the expansion, the total enthalpy and the entropy are conserved from Q to Q'. We shall say that Q and Q' are homologous.

This assumption leads to the consideration that the transition of a fluid particle from Q to Q' is so rapid that the viscosity effects are negligible. Under these conditions the calculations become easy.

First the distribution of Mach number M(Q) as a function of y/δ is /20
determined from the initial data. Since the expansion ratio $\frac{p_1}{p_0}$ is given, M'(Q') is deduced for the downstream location. Then, with the help of the continuity equation, the correspondence (Q, Q') remains to be determined; in other words, to every tube element of stream of thickness dy around Q, corresponds, through the isentropic expansion, an element dy' around Q such that

$$\rho u \, dy = \rho' u' \, dy'$$

But since the expansion is an isentropic one we have

$$\frac{\rho u}{\rho' u'} = \frac{\Sigma(M')}{\Sigma(M)}$$

Σ being the function $\frac{A}{A_0}$ of the Isentropic Expansion Tables, from which

$$y'(Q') = \int_0^{y(Q)} \frac{\Sigma(M')}{\Sigma(M)} \, dy$$

This relation yields the sought correspondence (Q, Q').

We have in this way the distribution M(y'), from which $\frac{u}{u_1}(y')$, $\frac{\rho}{\rho_1}(y')$, and consequently the boundary layer profile after the expansion are determined. The numerical calculations that we have performed in Ref. 3, for initial boundary layers having profiles of 1/5 of 1/7 powers, have shown us that we can express the momentum thickness δ^{**} after expansion, as a function of δ_0^{**} before expansion, regardless of M_0, by the relation

$$\frac{\rho_1 u_1 \delta^{**}}{\rho_0 u_0 \delta_0^{**}} = 1 - \frac{1}{2}\sqrt{\frac{M_1^2}{M_0^2} - 1}$$

Necessary Experimental Verifications

The preceding theory is based on a simplification of the general equations, where the turbulent mixture coefficients ε_t and λ_t are unknown.

With this scheme the problem is reduced to obtaining knowledge of the parameter σ and to verifying the fact that the temperature distribution in the mixture zone satisfies the assumption $P_{rt} = 1$.

The values of σ are well known from the incompressible regime up to Mach 2.5, approximately. Beyond this, the experimental results show more and more deviations as the Mach number increases. There are, however, very little data on the temperature distribution of a mixture layer. /21

Research is being conducted at the O.N.E.R.A. on these topics, under the direction of M. Sirieix. This research makes special use of interferometry, whose analysis was made accurate and automatic by the work of J. L. Solignac (Ref. 6).

Conclusion

The method which has been propounded in a few of its basic elements allows us from now on to predict the flow behavior at the base of a supersonic body, such as the supersonic transport plane Concorde, whose design studies have just been initiated.

More research is still necessary to extend its applications to the hypersonic field.

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